Your Signature ____________________

Instructions:
1. For writing your answers use both sides of the paper in the answer booklet.

2. Please write your name on every page of this booklet and every additional sheet taken.

3. If you are using a Theorem/Result from HW/class then please state the result clearly and verify the hypotheses of the same.

4. **Maximum time is 3 hours and Maximum Possible Score is 100.**

Score

Q.No.	**Alloted Score**	**Score**
1.	15	
2.	15	
3.	25	
4.	10	
5.	20	
6.	20	
Total	100	

Number of Extra sheets attached to the answer script: ____________________

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables on it.

   (a) (7 points) Find $\mathbb{P}(X_n \geq n \text{ i.o.})$ if $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 1$ for all $i \geq 1$.

   (b) (8 points) Show that for any $x \in \mathbb{R}$, $\mathbb{P}(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \leq x) \in \{0, 1\}$ if $\{X_n\}_{n\geq 1}$ are mutually independent.

2. Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be two probability spaces. Let $\{A_n\}_{n\geq 1}$ be a sequence of events in $\mathcal{F}_1$ and $\{B_n\}_{n\geq 1}$ be a sequence of events in $\mathcal{F}_2$ such that $A_n \times B_n$ are disjoint and $\bigcup_{n=1}^{\infty} A_n \times B_n = A \times B$ for some $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$.

(a) (10 points) For $\omega_1 \in \Omega_1$ show that $1_A(\omega_1)\mathbb{P}_2(B) = \sum_{n=0}^{\infty} 1_{A_n}(\omega_1)\mathbb{P}_2(B_n)$.

(b) (5 points) Show that $\mathbb{P}_1(A)\mathbb{P}_2(B) = \sum_{n=0}^{\infty} \mathbb{P}_1(A_n)\mathbb{P}_2(B_n)$.

: No credit for answer that assumes product measure exists.

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.Let $X, \{X_n\}_{n\geq 1}$ be a sequence of random variables on it.

   (a) (5 points) Show that $\limsup_{n\to\infty} X_n$ is also a random variable.

   (b) (5 points) Is it always true that $\mathbb{E}[\limsup_{n\to\infty} X_n] \geq \limsup_{n\to\infty} \mathbb{E}[X_n]$?

   (c) (7 points) Suppose $X_n \sim \text{Uniform}[0, n]$ then is it true that $\mathbb{P}_n = \mathbb{P} \circ X_n^{-1}$ is a tight sequence of Probability measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$?

   (d) (8 points) Suppose $\mathbb{P}(X_n = 1) = \frac{1}{n} = 1 - \mathbb{P}(X_n = 0)$. Let $Z_n = X + X_n$. Prove that $Z_n \stackrel{w}{\to} X$ as $n \to \infty$.

4. (10 points)Let $a_n = \exp(-n)\sum_{k=0}^{n}\frac{n^k}{k!}$. Using the Central Limit Theorem evaluate $\lim_{n\to\infty} a_n$.

5. Suppose $\{X_n\}_{n\geq 1}$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) (10 points) Show that for all $\epsilon > 0$

$$\mathbb{P}(\sup_{m\geq n} \mid X_m - X_n \mid > \epsilon) \to 0 \text{ as } n \to \infty$$

$$\Longrightarrow$$

$$\sup_{m\geq n} \mathbb{P}(\mid X_m - X_n \mid > \epsilon) \to 0 \text{ as } n \to \infty.$$

   (b) (10 points) Is the converse of (a) true ?

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{X_n : n \geq 1\}$ be identically distributed random variables on it with $\mathbb{E}(\mid X_1 \mid) < \infty$. Let for each $n \geq 1$, $M_n = \max\{X_1, X_2, \ldots X_n\}$. Show that

(a) (5 points) Show that $\int_{\sqrt{n}}^{\infty} \mathbb{P}(\mid X_1 \mid > t)dt \to 0$ as $n \to \infty$.

(b) (5 points) Show that $\mathbb{P}(\mid M_n \mid > t) \leq n\mathbb{P}(\mid X_1 \mid > t)$ for all $t \geq 0$.

(c) (10 points) Show that $\lim_{n\to\infty} \frac{\mathbb{E}[|M_n|]}{n} \to 0$.